Gorter-Mellink scale, and critical velocities in liquid–helium-II counterflow

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(Received 29 April 1974)

It is found that, in liquid–helium-II pure counterflow, the dimensionless number $\mathfrak{A} = \rho_s A\bar{w}l$ (where A is the Gorter-Mellink constant, $\bar{w}$ is the mean relative velocity between the two fluids, and l is a characteristic dimension of the geometry) scales the mutual friction between the two fluids. Critical relative velocities in cylindrical channels are shown to be correlated by the rule $\mathfrak{A}_c \equiv \rho_s A\bar{w}_c \pi d \simeq 1$.

I. INTRODUCTION

In a recent paper, Childers and Tough[1] measured the critical heat flux in pure counterflow (zero net mass flux) in a cylindrical tube. In view of the hysteresis, sensitivity to vibrations, and the condition of the walls that characterize the subcritical-supercritical transition,[2] it becomes necessary to specify the manner in which the critical velocity is defined. In the case of the data of Childers and Tough,[1] the measured critical velocity corresponded to the lowest heat flux that could support the supercritical regime. In practice this is measured by observing the temperature difference between the ends of the counterflow channel as the heat flux is decreased from a supercritical value to zero and observing the end of the supercritical region. In fact, such a definition is the only one that has been found to lead to reproducible results.

For a long cylindrical counterflow channel, the product of the critical relative velocity, defined in this manner, and the diameter of the channel is found to be only a function of temperature.[1-3] To illustrate the point, the data of Childers and Tough[1] and of Brewer and Edwards[2] are plotted in Fig. 1. The incomplete error bars on the Brewer-Edwards data correspond to the size of the symbols used in their plot. Error intervals that have been specified explicitly by the experimenters are depicted in the standard fashion. Unfortunately, it is not possible to interpret the wealth of other data on the critical velocity in cylindrical channels because insufficient attention seems to have been paid to the important effect of mechanical vibrations.

The early theoretical attempts by Landau[4] to compute a critical velocity on the basis of the dispersion relation of the elementary excitation in liquid helium II, even though of historical significance, yielded velocities which were too high. Subsequent work by Feynman,[5] based on a model in which the critical velocity is computed as the minimum velocity required to produce a quantized vortex line, yielded numerical estimates which were much closer to observed values. That theoretical estimate, however, expressed the critical velocity as a function of the geometry in terms of fundamental microscopic quantities and it is not clear how the temperature dependence of the observed values could be incorporated into such a theory. A later theory by Vinen,[6] of the mutual friction mechanism that had been postulated by Gorter and Mellink,[11] based on a model of dynamic equilibrium between vortex line generation and annihilation has been quoted[1,3] as successful in interpreting critical velocity data. Unfortunately, the number of empirical parameters in the theory that must be determined by experiment in order to fit the data makes it difficult to find a means of determining the extent of the validity of the theory.

An attempt will be made in this paper to show that it is possible to determine the temperature and dimensional dependence of the critical velocity in cylindrical channels in pure counterflow from the form of the two-fluid equations of motion. Earlier attempts along these lines[7,8] have tried to include normal-fluid viscous effects. There exists experimental evidence,[9,10] however, that the transition to supercritical flow does not affect pressure differences and is therefore probably uncoupled from the normal fluid. These experiments, however, were done without due care as regards mechanical vibrations and may therefore be suspect. In addition, it will be shown in this paper that the proper scaling of the critical velocity is obtained without any consideration of viscous mechanisms.

II. GORTER-MELLINK SCALE

If we assume that the phenomenological friction term introduced by Gorter and Mellink[11] is appropriate for steady supercritical flow and if we neglect the higher viscosity terms, we can write the equations of motion (for supercritical flow) as follows:

$$\nabla \cdot \rho_n \vec{u}_n = -(\nabla \cdot \rho_s \vec{u}_s) = \Gamma, \tag{1}$$

$$(\vec{u}_s \cdot \nabla)\vec{u}_s = -\nabla\mu + A\rho_n w^2 \vec{w}, \tag{2}$$

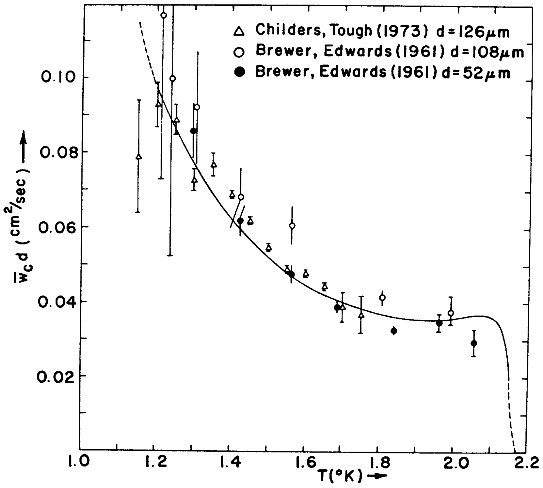

FIG. 1. Mean critical relative velocity $\bar{w}_c$, times the diameter d, for a cylindrical channel, vs temperature. For smooth line see text.

$$(\vec{u}_n\cdot\nabla)\vec{u}_n = -\nabla\mu - A\rho_s w^2\vec{w} - \frac{s\rho}{\rho_n}\nabla T - \tfrac{1}{2}\nabla w^2 - \vec{w}(\Gamma/\rho_n) + \nu_n\nabla^2\vec{u}_n\,, \tag{3}$$

where

$$\nabla\mu = -s\nabla T + (1/\rho)\nabla p - \tfrac{1}{2}(\rho_n/\rho)\nabla w^2 \tag{4}$$

is the gradient of the chemical potential,

$$\vec{w} = \vec{u}_n - \vec{u}_s \tag{5}$$

is the relative velocity between the two fluids, and

$$\nu_n = \eta_n/\rho_n \tag{6}$$

is the kinematic viscosity of the normal fluid. Subtracting Eq. (2) from Eq. (3) we have

$$(\vec{u}_n\cdot\nabla)\vec{u}_n - (\vec{u}_s\cdot\nabla)\vec{u}_s + \tfrac{1}{2}\nabla w^2 + \vec{w}(\Gamma/\rho_n) = -\rho A w^2\vec{w} - (s\rho/\rho_n)\nabla T + \nu_n\nabla^2\vec{u}_n \tag{7}$$

or, eliminating $\vec{u}_s$ in terms of Eq. (5), we have:

$$(\vec{w}\cdot\nabla)\vec{u}_n + (\vec{u}_n\cdot\nabla)\vec{w} + \vec{w}\times(\nabla\times\vec{w}) + \vec{w}(\nabla\cdot\vec{u}_n) + \vec{w}(\vec{u}_n\cdot\nabla\ln\rho_n) = -\rho A w^2\vec{w} - (s\rho/\rho_n)\nabla T + \nu_n\nabla^2\vec{u}_n\,, \tag{8}$$

where Γ has been eliminated in terms of Eq. (1).

To express Eq. (8) in dimensionless form, we make the following substitutions:

$$\vec{w} = \bar{w}\vec{w}^*\,, \tag{9a}$$

$$\vec{u}_n = (\rho/\rho_s)\bar{w}\vec{u}_n^*\,, \tag{9b}$$

$$T = T_0 T^*\,, \tag{9c}$$

$$\rho_n = \rho_{n0}\rho_n^*\,, \tag{9d}$$

$$\nabla = (1/l)\nabla^*\,, \tag{9e}$$

where $\bar{w}$ is the mean relative velocity between the two fluids and given, in terms of the mean heat flux $\bar{q}$, by

$$\bar{w} = \bar{q}/\rho_s s T\,, \tag{10}$$

l is the characteristic dimension of the geometry, T_0 and ρ_{n0} are evaluated at ambient conditions, and starred quantities are dimensionless. In writing Eq. (9b), the assumption of no net mass flux has been introduced (pure counterflow), i.e.,

$$\rho_n\bar{u}_n + \rho_s\bar{u}_s = 0\,,$$

where $\bar{u}_n$ and $\bar{u}_s$ are the (algebraic) values of the velocities averaged over the area of the channel. With this condition and using

$$\bar{w} = \bar{u}_n - \bar{u}_s$$

we can express $\bar{u}_n$ in terms of $\bar{w}$ to yield Eq. (9b). With these substitutions, Eq. (8) can be written as

$$(\vec{w}^*\cdot\nabla^*)\vec{u}_n^* + (\vec{u}_n^*\cdot\nabla^*)\vec{w}^* + \vec{w}^*(\nabla^*\cdot\vec{u}_n^*) + \vec{w}^*(\vec{u}_n^*\cdot\nabla^*\ln\rho_n^*) + \frac{\rho_s}{\rho}\vec{w}^*\times(\nabla^*\times\vec{w}^*) = -(\rho_s A\bar{w}l)w^{*2}\vec{w}^* - \frac{s\rho_s T_0}{\rho_n\bar{w}^2}\nabla^* T^* + \frac{\nu_n}{\bar{w}l}\nabla^{*2}\vec{u}_n^*\,. \tag{11}$$

Some of the dimensionless coefficients can be expressed in a more familiar form. Let

$$\mathcal{R} \equiv \frac{\bar{u}_n l}{\nu_n} = \frac{\rho}{\rho_s}\frac{\bar{w}l}{\nu_n} \tag{12}$$

be the Reynolds number for the normal fluid. Since the speed of second sound u_{II} is given by

$$u_{\mathrm{II}}^2 = \rho_s s^2 T/c\rho_n\,, \tag{13}$$

we can define

$$M \equiv \bar{w}/u_{\mathrm{II}}\,, \tag{14}$$

the Mach number of the mean relative velocity with respect to the speed of second sound. Then, Eq. (11) becomes

$$(\vec{w}^*\cdot\nabla^*)\vec{u}_n^* + (\vec{u}_n^*\cdot\nabla^*)\vec{w}^* + \vec{w}^*(\nabla^*\cdot\vec{u}_n^*) + \frac{\rho_s}{\rho}\left[\vec{w}^*\times(\nabla^*\times\vec{w}^*) - \frac{1}{\mathcal{R}}\nabla^{*2}\vec{u}_n^*\right] = -\mathcal{A}\,w^{*2}w^* - \frac{\alpha}{M^2}\nabla^* T^* - w^*(u_n^*\cdot\nabla^*\ln\rho_n^*)\,, \tag{15}$$

where α is the ratio of the heat capacity to the entropy, or

$$\alpha \equiv \frac{c}{s} = \frac{T}{s}\frac{\partial s}{\partial T} \simeq 5.6 \tag{16}$$

and

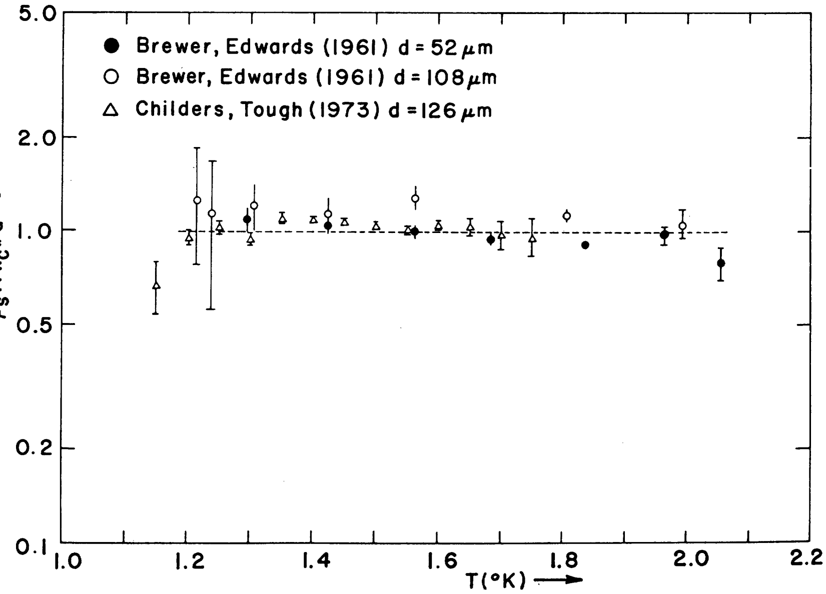

FIG. 2. Critical dimensionless Gorter-Mellink scale $\mathcal{a}_c \equiv \rho_s A \bar{w}_c \pi d$ vs temperature.

$$\mathcal{a} \equiv \rho_s A \bar{w} l \tag{17}$$

is the dimensionless number scaling the mutual friction term. The asymmetric dependence on ρ_s seems to hint that $\bar{w}$ might not be the fundamental quantity but rather $\bar{q}$, the mean heat flux. Using Eq. (10) we then have

$$\mathcal{a} \equiv \frac{A\bar{q}l}{sT} . \tag{17'}$$

III. CRITICAL VELOCITY

To the extent that the number $\mathcal{a}$ is the scale of the mutual friction term, equivalent flow situations should correspond to a certain value of $\mathcal{a}$. In particular, if the number $\mathcal{a}$ is evaluated for $\bar{w}=\bar{w}_c$, the critical relative velocity, a critical value of $\mathcal{a}$ should exist, independent of temperature, i.e.,

$$\mathcal{a}_c \equiv \rho_s A \bar{w}_c l = \text{const.}$$

To test this hypothesis, $\mathcal{a}$ was computed for the data presented in Fig. 1. The Gorter-Mellink A was evaluated with the aid of the empirical formula[12]

$$\log_{10} A(T) = 1.10 + 3.12 \log_{10} T + \frac{0.0076}{1 - T/T_\lambda} , \tag{18}$$

while the choice of $l = \pi d$, the perimeter of the tube, was made for reasons that will become evident. In Fig. 2 the quantity $\mathcal{a}_c = \rho_s A \bar{w}_c \pi d$ was plotted versus temperature. It can be seen that the data support the hypothesis. It is intriguing that the particular value of $\mathcal{a}_c$ could very well be equal to unity. The smooth line in Fig. 1 is a plot of

$$\bar{w}_c d = 1/\pi \rho_s A .$$

IV. CONCLUSION

In conclusion, it seems that the dimensionless number $\mathcal{a} = \rho_s A \bar{w} l$ is the appropriate scale for the Gorter-Mellink mutual friction in pure counterflow, which can be obtained by writing the two-fluid equations in a dimensionless form. In addition, the data for pure counterflow in cylindrical channels support the empirical rule for the critical velocity as given by $\rho_s A \bar{w}_c \pi d \approx 1$.

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